

UNIFORMITY OF THE MAGNETIC FIELD IN A HELMHOLTZ COIL CONFIGURATION

1. On-Axis Case

The magnitude of the magnetic field $|B|$ a distance x from the center of a single coil of radius a with N turns and carrying a current I is (the derivation is given below for the general off-axis case)

$$|B| = \frac{\mu_0 N I a^2}{2(x^2 + a^2)^{3/2}}, \quad (1)$$

and for the current direction shown in Figure 1 is directed toward the negative x direction (right hand rule).

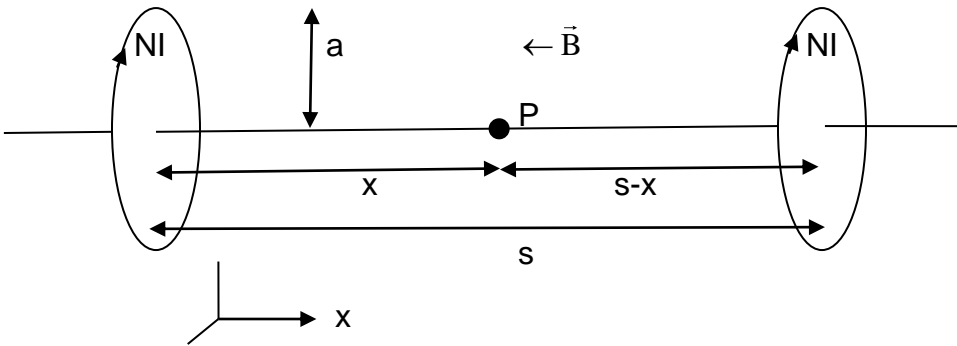


Figure 1

The planes of the coils are perpendicular to the axis. Define the separation of the two Helmholtz coils to be s and the distance of an on-axis point P from each of the coils to be x and $(s - x)$ (see Figure 1). The combined field at point P is then

$$|B| = \frac{\mu_0 N I a^2}{2} \left\{ \frac{1}{(x^2 + a^2)^{3/2}} + \frac{1}{[(s-x)^2 + a^2]^{3/2}} \right\}. \quad (2)$$

The condition for $|B|$ to vary least with position along the x -axis is that $(d|B|/dx)$ be a minimum so that $(d^2|B|/dx^2) = 0$:

$$\begin{aligned}\frac{d|B|}{dx} &= B_0 \left\{ -\frac{3}{2}(x^2 + a^2)^{-5/2} (2x) - \frac{3}{2}[(s-x)^2 + a^2]^{-5/2} [2(s-x)](-1) \right\} \\ &= 3B_0 \left\{ -x(x^2 + a^2)^{-5/2} + (s-x)[(s-x)^2 + a^2]^{-5/2} \right\}\end{aligned}\quad (3)$$

where

$$B_0 \equiv \frac{\mu_0 N I a^2}{2} . \quad (4)$$

Then

$$\frac{d^2|B|}{dx^2} = 3B_0 \left\{ \begin{aligned} &-\left(x^2 + a^2\right)^{-5/2} - x\left(\frac{-5}{2}\right)\left(x^2 + a^2\right)^{-7/2} (2x) - \left[(s-x)^2 + a^2\right]^{-5/2} \\ &-\left(\frac{5}{2}\right)(s-x)\left[(s-x)^2 + a^2\right]^{-7/2} [2(s-x)](-1) \end{aligned} \right\} \quad (5a)$$

$$= 3B_0 \left\{ \begin{aligned} &-\left(x^2 + a^2\right)^{-5/2} + 5x^2\left(x^2 + a^2\right)^{-7/2} \\ &-\left[(s-x)^2 + a^2\right]^{-5/2} 5(s-x)^2\left[(s-x)^2 + a^2\right]^{-7/2} \end{aligned} \right\} = 0 \quad (5b)$$

Insertion of the condition $x = s / 2$ into eq. (5b) yields

$$\begin{aligned}-\left(\frac{s^2}{4} + a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4} + a^2\right)^{-7/2} - \left(\frac{s^2}{4} + a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4} + a^2\right)^{-7/2} &= 0 \\ \Rightarrow -\left(\frac{s^2}{4} + a^2\right)^{-5/2} + 5\left(\frac{s^2}{4}\right)\left(\frac{s^2}{4} + a^2\right)^{-7/2} &= 0,\end{aligned}\quad (6)$$

and multiplying through by $\left(\frac{s^2}{4} + a^2\right)^{7/2} \neq 0$ gives the desired answer:

$$-\left(\frac{s^2}{4} + a^2\right) + 5\left(\frac{s^2}{4}\right) = 0 \Rightarrow s = \pm a . \quad (7)$$

THUS THE OPTIMUM SEPARATION s BETWEEN THE COILS IS THE COIL RADIUS a .

The minimum value of $\left(\frac{d|B|}{dx}\right)$ is obtained by inserting $s = a$ and $x = s / 2$ into eq. (3):

$$\left.\frac{d|B|}{dx}\right|_{\min} = 3B_0 \left\{ -\frac{a}{2} \left(\frac{5a^2}{2}\right)^{-5/2} + \frac{a}{2} \left(\frac{5a^2}{2}\right)^{-5/2} \right\} = 0. \quad (8)$$

The values of $|B|$ at the center of each coil and half way between the coils are

$$x = 0: \quad |B| = \frac{B_0}{a^3} \left(1 + \frac{1}{2^{3/2}}\right) = \frac{(1.3536)B_0}{a^3} \quad (9)$$

and

$$x = a / 2: \quad |B| = \frac{B_0}{a^3} \left[\frac{2}{(5/4)^{3/2}} \right] = \frac{(1.4311)B_0}{a^3}, \quad (10)$$

yielding a ratio of $(1.4311/1.3536) = 1.057$. Thus the on-axis magnetic field between the coils is constant to within about 5.7%.

A GNU Octave plot of $|B|$ vs. x/a computed from equation (2) is included in Figure 3 below as a special case ($h = 0$) for positions a distance h off-axis. The on-axis value of $|B|$ varies by less than 1% for $0.8 \geq x/a \geq 0.2$ and by less than 0.1% for $0.6 \geq x/a \geq 0.4$.

2. Off-Axis Case

Use the Biot–Savart law

$$|d\vec{B}| = \left(\frac{\mu_0 NI}{4\pi |r^3|} \right) d\vec{s} \times \vec{r}, \quad (11)$$

where $|ds| = a d\theta$. Consider a point that is an orthogonal distance h from the central axis that passes through the center of each coil (see Figure 2), and let a , NI , and x be as above.

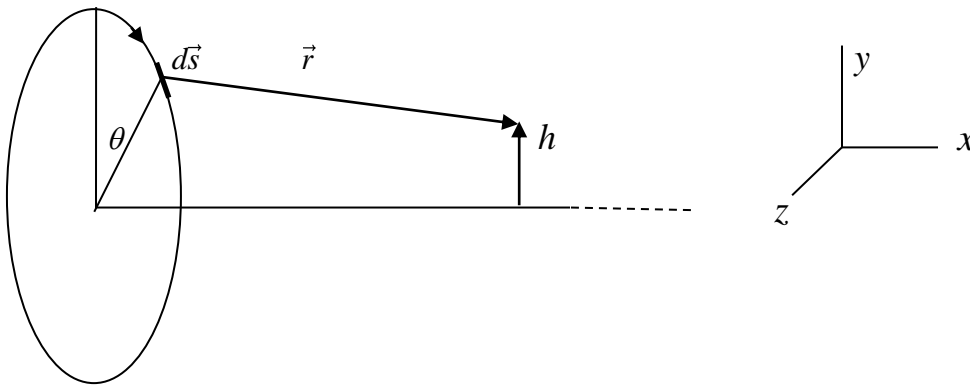


Figure 2

The components of $d\vec{s}$ are

$$\left. \begin{aligned} ds_x &= 0 \\ ds_y &= -ds \sin \theta = -a \sin \theta d\theta \\ ds_z &= -ds \cos \theta = -a \cos \theta d\theta \end{aligned} \right\} \quad (12a)$$

and the components of \vec{r} are

$$\left. \begin{aligned} r_x &= x \\ r_y &= h - a \cos \theta \\ r_z &= +a \sin \theta \end{aligned} \right\} \quad (12b)$$

Thus

$$\begin{aligned}
|r^3| &= \left[x^2 + a^2 \sin^2 \theta + (h^2 + a^2 \cos^2 \theta - 2ah \cos \theta) \right]^{3/2} \\
&= \left[x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
d\vec{s} \times \vec{r} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & -a \sin \theta d\theta & -a \cos \theta d\theta \\ x & h - a \cos \theta & a \sin \theta \end{vmatrix} \\
&= \hat{i} \left[-a^2 \sin^2 \theta d\theta + a \cos \theta (h - a \cos \theta) d\theta \right] - \hat{j} \left[ax \cos \theta d\theta \right] + \hat{k} \left[ax \sin \theta d\theta \right]
\end{aligned} \tag{14}$$

With $B_0 = \mu_0 N I a^2 / 2$ as before the components of the field from the leftmost coil in Fig.1 are therefore

$$B_x = B_0 \int_0^{2\pi} \frac{-a^2 d\theta}{\left[x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}} + \int_0^{2\pi} \frac{ah \cos \theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}; \tag{15a}$$

$$B_y = B_0 \int_0^{2\pi} \frac{ax \cos \theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}; \tag{15b}$$

$$B_z = B_0 \int_0^{2\pi} \frac{ax \sin \theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}. \tag{15c}$$

The corresponding expressions for the rightmost coil are

$$B_x = B_0 \int_0^{2\pi} \frac{-a^2 d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}} + \int_0^{2\pi} \frac{ah \cos \theta d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}; \quad (16a)$$

$$B_y = B_0 \int_0^{2\pi} \frac{a(s-x) \cos \theta d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}; \quad (16b)$$

$$B_z = B_0 \int_0^{2\pi} \frac{a(s-x) \sin \theta d\theta}{\left[(s-x)^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}}. \quad (16c)$$

There are three integrals in eqs. (15) and (16) of which only one is simple:

$$B_z = B_0 \int_0^{2\pi} \frac{ax \sin \theta d\theta}{\left[x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}} = B_0 \int_0^{2\pi} \frac{ax d \cos \theta}{\left[x^2 + a^2 + h^2 - 2ah \cos \theta \right]^{3/2}} = 0. \quad (17)$$

The integrals for $h = 0$ are

$$B_x = B_0 \int_0^{2\pi} \frac{-a^2 d\theta}{\left[x^2 + a^2 \right]^{3/2}} = \frac{-a^2 B_0 2\pi}{\left[x^2 + a^2 \right]^{3/2}}, \quad (18)$$

$$B_y = B_0 \int_0^{2\pi} \frac{ax \cos \theta d\theta}{\left[x^2 + a^2 \right]^{3/2}} = \frac{B_0 ax}{\left[x^2 + a^2 \right]^{3/2}} \int_0^{2\pi} \cos \theta d\theta = 0. \quad (19)$$

Equations (4) and (16) yield eq. (1). For $h \neq 0$ the integrals for B_x and B_y in eqs. (16a) and (16b) are elliptical but are easily computed numerically. Plots of four magnetic field properties vs. distance x from the leftmost coil are shown in Figure 3 below for four values of $h/a = 0; 0.2; 0.4; 0.6$. The plotted properties are

- (1) Magnitude of the field $|B|$;
- (2) x component of $B = B_x$;
- (3) y component of $B = B_y$;
- (4) Angle (in degrees) between B_y and $B_x = \text{atan}(|B_y|/|B_x|)$.

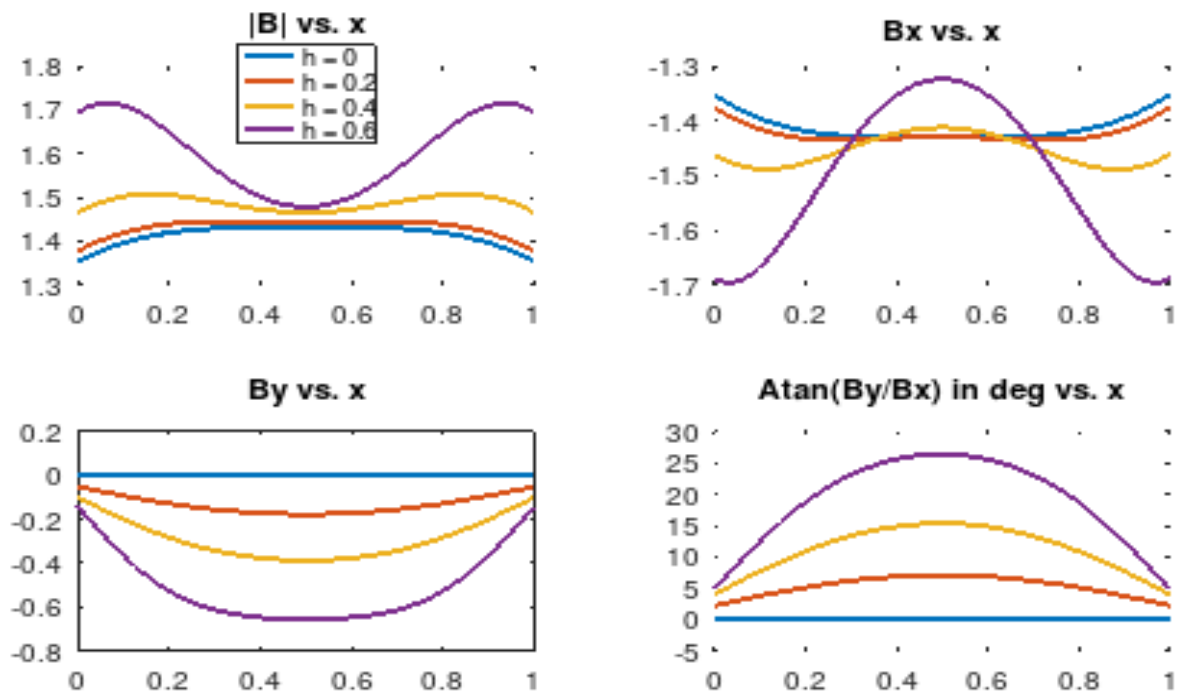


Figure 3